The Parametrized Post–Newtonian Gravitational Redshift

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ABSTRACT

A derivation of the gravitational redshift effect to order c^{-4} is presented. The calculation is performed within the framework of the parametrized post–Newtonian formalism for analyzing metric theories of gravity, which includes corrections to second–order in the Newtonian potential, gravitomagnetic contributions, and preferred-frame terms. We briefly discuss how to generalize our results to include possible violations of local Lorentz invariance or local position invariance which can arise in nonmetric theories. Our results are useful for analyzing possible new redshift experiments which may be sensitive to second-order effects, such as a close solar flyby mission.

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I. Introduction

The gravitational redshift effect is the observed shift to a lower frequency of an oscillator near a massive body relative to its frequency at infinity. This is a fundamental result of the Einstein Equivalence Principle (EEP), upon which general relativity and all other metric theories of gravity are based.[1] If the EEP is valid, then the laws of physics governing the operation of an oscillator or a clock should be locally Lorentz-invariant and position-invariant in a gravitational field. By adopting only these two requirements, it is possible to derive the redshift effect to first-order in the Newtonian potential without specifying a particular theory of gravity (for example, see Ref. [2]).

To order c^{-2}, the frequency shift of a photon propagated between two points $\vec{x}_1$ and $\vec{x}_2$ is given in an inertial reference frame by the expression

$$\begin{aligned} f_2 = f_1[1 - \hat{n}\cdot(\vec{v}_2 - \vec{v}_1)/c - (1/2)(v_1^2 - v_2^2)/c^2 - (\hat{n}\cdot\vec{v}_1)(\hat{n}\cdot\vec{v}_2)/c^2 \\ + (\hat{n}\cdot\vec{v}_1)^2/c^2 - (U_1 - U_2)/c^2] \end{aligned} , \qquad (1.1)$$

where $\vec{v}_1$ is the velocity of the emitter at $\vec{x}_1$, $\vec{v}_2$ is the velocity of the receiver at $\vec{x}_2$ $\hat{n}$ is a unit vector pointing from $\vec{x}_1$ to $\vec{x}_2$, and U_1 and U_2 are the total Newtonian gravitational potentials at each point, respectively, defined positively. Eq. (1.1) is consistent with the EEP to first-order in U. Any metric theory of gravity, such as general relativity, must make the same prediction to this order, provided the theory yields the correct Newtonian equation of motion.

The first-order prediction has been tested to highest precision in a 1976 NASA experiment called Gravity Probe A (GP-A), in which a hydrogen maser oscillator was flown on a Scout rocket in the gravitational field of the Earth.[3] Additional spacecraft experiments have been performed at Saturn[4] and in the solar gravitational field,[5] but with less stable crystal oscillators. A close solar probe mission has been studied by NASA for many years, in which a spacecraft would fly by the Sun at a heliocentric distance of only 4 solar radii (for a recent review, see Ref. [6]). Similar missions have been considered by the European Space Agency (ESA) and by the Russian Institute for Space Research (IKI). If an atomic frequency standard, such as a hydrogen maser oscillator, were included on the spacecraft, then it might be possible to test the redshift effect to second-order in the Newtonian potential of the Sun at an interesting level of precision. At second-order, the experiment would test not only the EEP, but also specific theories of gravity. A small group of scientists has recently been funded by NASA to investigate this possibility. A group at JPL is performing a mission simulation and a detailed covariance analysis. A second group at the Smithsonian Center for Astrophysics is investigating requirements on the maser flight unit. At this point, however, NASA has made no definite commitment to proceed with the mission.

In a previous study, general relativistic effects on the equation of motion of the spacecraft were modeled to second-order in the PPN formalism, but the redshift was modeled to only first-order.[7] In this report, we present a derivation of the gravitational redshift to order c^{-4} in support of the new study. We will adopt the parametrized post-Newtonian (PPN) framework for analyzing metric theories of gravity. However, we will restrict our analysis to semi-conservative theories of gravity, in which the PPN parameters $\{\alpha_3, \zeta_1, \zeta_2, \zeta_3, \zeta_4\}$ are identically zero. Furthermore, we will assume a single, stationary body whose center of mass is at rest, and which is rotating slowly enough that we may also assume nearly spherical symmetry; appropriate assumptions for analyzing a close solar flyby mission. Unlike the first-order calculation, we will see that it is necessary to consider the photon equation of motion at this level of accuracy. Although our main interest is in PPN effects on the redshift, we will briefly discuss how to include possible violations of local Lorentz invariance and local position invariance.

The remainder of this paper is organized as follows. In the next Section, we will derive a general form for the redshift which includes contributions from the photon wave vector. In Section III, we will integrate the photon equation of motion. Our final results are presented in Section IV, and conclusions in Section V. For convenience, units in which $G = c = 1$ will be used. Greek indices range over $0, 1, 2, 3$, whereas latin indices range over $1, 2, 3$. Partial derivatives are denoted by a comma, and covariant derivatives with respect to the metric connection by a semicolon.

II. The Measured Frequency Shift

For metric theories of gravity, the frequency f_m of a photon measured by an observer at a point (t, x^i) can be found by projecting the photon wave-vector k onto the observer's four-velocity u at that point:

$$\omega_m = -k_\mu u^\mu = -(k_0 u^0 + k_i u^i), \tag{2.1}$$

where $\omega_m = 2\pi f_m$. We can expand the components of k about their flat spacetime values according to

$$k_0(x) = -\omega[1 - \hat{k}_0(x)], \tag{2.2a}$$

$$k_i(x) = \omega[\hat{n}_i + \hat{k}_i(x)], \tag{2.2b}$$

where $\hat{n}_i$ is a unit vector along the direction of propagation. Eq. (2.1) then becomes

$$\omega_m = \omega u^0[1 - \hat{k}_0(\underset{\sim}{x}) - \underset{\sim}{\hat{n}} \cdot \underset{\sim}{v} - \underset{\sim}{\hat{k}}(\underset{\sim}{x}) \cdot \underset{\sim}{v}], \tag{2.3}$$

where $\underset{\sim}{v}$ is the observer's three-velocity. Eq. (2.3) can be used to calculate the ratio of the frequencies which would be measured at two points $x_1 = (t_1, x_1^i)$ and $x_2 = (t_2, x_2^i)$. Expanded to order v^4, the result is

$$
\begin{aligned}
\frac{\omega'(\mathbf{x}_2)}{\omega(\mathbf{x}_1)} = \frac{\omega^{\circ}(\mathbf{x}_2)}{\omega^{\circ}(\mathbf{x}_1)} \Big\{ & 1 - \hat{\mathbf{n}} \cdot (\mathbf{v}_2 - \mathbf{v}_1) \\
& - \hat{\mathbf{n}} \cdot (\mathbf{v}_2 - \mathbf{v}_1) \left[\hat{\mathbf{n}} \cdot \mathbf{v}_1 + (\hat{\mathbf{n}} \cdot \mathbf{v}_1)^2 + (\hat{\mathbf{n}} \cdot \mathbf{v}_1)^3 \right] \\
& - \left[\hat{\mathbf{k}}(\mathbf{x}_2) \cdot \mathbf{v}_2 - \hat{\mathbf{k}}(\mathbf{x}_1) \cdot \mathbf{v}_1 \right] \\
& + 2 (\hat{\mathbf{n}} \cdot \mathbf{v}_1)(\hat{\mathbf{k}}(\mathbf{x}_1) \cdot \mathbf{v}_1) - \hat{\mathbf{k}}(\mathbf{x}_2) \cdot \mathbf{v}_2 (\hat{\mathbf{n}} \cdot \mathbf{v}_1) \\
& - \hat{\mathbf{k}}(\mathbf{x}_1) \cdot \mathbf{v}_1 (\hat{\mathbf{n}} \cdot \mathbf{v}_2) \Big\} . \qquad (2.4)
\end{aligned}
$$

where we have neglected contributions from $\hat{k}_0$ (to be justified in the next Section). Eq. (2.4) provides a general form for the measurable frequency shift. To make Eq. (2.4) more useful, we will have to provide specific expressions for u^0 and the perturbed wave-vector components $(\hat{k}_0, \hat{k}_i)$.

With proper time defined by $d\tau^2 = -g_{\mu\nu}dx^\mu dx^\nu$, then

$$u^0 = \frac{dx^0}{d\tau} = (-g_{00})^{-1/2}\left(1 + 2\frac{g_{0j}}{g_{00}}v^j + \frac{g_{ij}}{g_{00}}v^i v^j\right)^{-1/2}, \tag{2.5}$$

Under the assumptions stated in Section I, the PPN metric components are given by

$$\begin{aligned} g_{00} &= -1 + 2U - 2\beta U^2 - (\alpha_1 - \alpha_2)w^2 U - \alpha_2 U(\underset{\sim}{w} \cdot \underset{\sim}{\hat{r}})^2 \\ &\quad + \frac{1}{2}\alpha_1 \underset{\sim}{w} \cdot (\underset{\sim}{\hat{r}} \times \underset{\sim}{J})/r^2, \end{aligned} \tag{2.6a}$$

$$g_{0j} = \frac{1}{4}\Delta\frac{(\underset{\sim}{\hat{r}} \times \underset{\sim}{J})_j}{r^2} - \frac{1}{2}(\alpha_1 - 2\alpha_2)Uw^j - \alpha_2(\underset{\sim}{\hat{r}} \cdot \underset{\sim}{w})U\hat{r}^j, \tag{2.6b}$$

$$g_{ij} = (1 + 2\gamma U)\delta_{ij}, \tag{2.6c}$$

where $\underset{\sim}{\hat{r}} = \underset{\sim}{x}/|\underset{\sim}{x}|$, $\Delta = 4\gamma + 4 + \alpha_1$, and $\underset{\sim}{w}$ is the possible preferred-frame velocity of the PPN coordinate system (for example, see Ref. [8]). The vector $\underset{\sim}{J}$ represents the rotational angular momentum of the body. Using these metric components in Eq. (2.5), we obtain to order c^{-4}

$$
\begin{aligned}
u^0 = {} & 1 + U + \frac{1}{2}v^2 + \left(\frac{3}{2} - \beta\right)U^2 + \left(\frac{3}{2} + \gamma\right)Uv^2 \\
& + \frac{3}{8}v^4 - \frac{1}{2}(\alpha_1 - \alpha_2 - \alpha_3)w^2 U - \frac{1}{2}\alpha_2 U(\mathbf{w}\cdot\hat{\mathbf{r}})^2 \\
& + \frac{1}{4}(\alpha_1 - 2\alpha_3)\,\mathbf{w}\cdot(\hat{\mathbf{r}}\times\mathbf{J})/r^2 + \frac{1}{4}A\,\mathbf{v}\cdot(\hat{\mathbf{r}}\times\mathbf{J})/r^2 \\
& - \frac{1}{2}(\alpha_1 - 2\alpha_2)(\mathbf{v}\cdot\mathbf{w})U - \alpha_2 U(\mathbf{v}\cdot\hat{\mathbf{r}})(\mathbf{w}\cdot\hat{\mathbf{r}}) .
\end{aligned}
\tag{2.7}
$$

This expression can be used to calculate the ratio $u^0(\underset{\sim}{x}_2)/u^0(\underset{\sim}{x}_1)$ in Eq. (2.4). The result is

$$
\begin{aligned}
\frac{U^0(\underset{\sim}{x}_2)}{U^0(\underset{\sim}{x}_1)}
&= 1 - (U_1 - U_2) - \tfrac{1}{2}(v_1^2 - v_2^2) \\
&\quad - \tfrac{3}{8}(v_1^4 - v_2^4) - \tfrac{1}{4} v_1^2 v_2^2 + \tfrac{1}{4} v_1^4 \\
&\quad - \left[(\tfrac{1}{2} - \beta) U_1^2 - (\tfrac{3}{2} - \beta) U_2^2 + U_1 U_2 \right] \\
&\quad - \left[(\tfrac{1}{2} + \gamma) U_1 v_1^2 - (\tfrac{3}{2} + \gamma) U_2 v_2^2 + \tfrac{1}{2}(U_1 v_2^2 + U_2 v_1^2) \right] \\
&\quad - \tfrac{1}{2} \Big\{ (\alpha_1 - \alpha_2 - \alpha_3) w^2 (U_1 - U_2) + \alpha_2 \left[U_1 (\underset{\sim}{w} \cdot \hat{\underset{\sim}{r}}_1)^2 - U_2 (\underset{\sim}{w} \cdot \hat{\underset{\sim}{r}}_2)^2 \right] \\
&\qquad + (\alpha_1 - 2\alpha_2) \left[(\underset{\sim}{v}_1 \cdot \underset{\sim}{w}) U_1 - (\underset{\sim}{v}_2 \cdot \underset{\sim}{w}) U_2 \right] \\
&\qquad + 2\alpha_2 \left[U_1 (\underset{\sim}{v}_1 \cdot \hat{\underset{\sim}{r}}_1)(\underset{\sim}{w} \cdot \hat{\underset{\sim}{r}}_1) - U_2 (\underset{\sim}{v}_2 \cdot \hat{\underset{\sim}{r}}_2)(\underset{\sim}{w} \cdot \hat{\underset{\sim}{r}}_2) \right] \Big\} \\
&\quad + \tfrac{1}{4} \Big\{ A \left[\underset{\sim}{v}_1 \cdot (\hat{\underset{\sim}{r}}_1 \times \underset{\sim}{J})/r_1^2 - \underset{\sim}{v}_2 \cdot (\hat{\underset{\sim}{r}}_2 \times \underset{\sim}{J})/r_2^2 \right] \\
&\qquad + (\alpha_1 - 2\alpha_3) \underset{\sim}{w} \cdot \left[(\hat{\underset{\sim}{r}}_1 \times \underset{\sim}{J})/r_1^2 - (\hat{\underset{\sim}{r}}_2 \times \underset{\sim}{J})/r_2^2 \right] \Big\}
\end{aligned}
\tag{2.8}
$$

III. Computation of the Photon Wave–Vector

It is convenient to define a tensor $h_{\mu\nu}$ such that

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \tag{3.1}$$

where $\eta = \mathrm{diag}(-1, +1, +1, +1)$ is the Minkowski tensor. Using this definition in the equation of motion for the photon wave–vector, given by

$$k_{\mu;\nu} k^{\nu} = 0, \tag{3.2}$$

results in the expression

$$\frac{dk_\mu}{d\lambda} = \frac{1}{2} h_{\beta\nu},_{\mu} k^{\beta} k^{\nu} + O(h^3 k^2), \tag{3.3}$$

where λ is an affine parameter along the photon trajectory, and indices are now raised and lowered with η. Expanding k_μ according to Eq. (2.2) and applying the condition $k_\mu k^\mu = 0$, we obtain from equations (2.6) and (3.3) the result

$$\frac{d\hat{k}_\mu}{dt} = (1+\gamma) U,_{\mu} + h_{0j},_{\mu} \hat{n}^j, \tag{3.4}$$

to necessary order, where t is coordinate time. Equation (3.4) can be integrated to obtain $\hat{k}_\mu(x)$. We note that $\hat{k}_0$ is zero at this order for no explicit time dependence of the source, which is why it was not included in Eq. (2.4).

To integrate Eq. (3.4) for $\hat{k}_i$, we assume a nearly spherically symmetric source such that $U(\underset{\sim}{x}) = M/r$; the expression for h_{0j} is dictated by Eq. (2.6b). The unperturbed path of the photon is a straight line with coordinates $\underset{\sim}{x}(t) = \underset{\sim}{\hat{n}}(t-t_e)+\underset{\sim}{x}_e$ as a function of t, where $(t_e, \underset{\sim}{x}_e)$ specifies the time and place of emission. To necessary order, this relation can be used to integrate Eq. (3.4). However, the constant of integration is specified by requiring that the orthogonal projection of the actual perturbed path satisfies

$$\left.\frac{dx_\perp^i}{dt}\right|_{t_e} = 0, \tag{3.5a}$$

$$x_\perp^i = x^i - \hat{n}^i(\underset{\sim}{\hat{n}} \cdot \underset{\sim}{x}), \tag{3.5b}$$

i.e., that upon emission the photon propagates initially in the direction $\underset{\sim}{\hat{n}}$, where

$$\frac{dx^i}{dt} = [\hat{n}^i - 2(1+\gamma)U\hat{n}^i + \hat{k}^i], \tag{3.6}$$

to the required order. After evaluating several integrals and collecting terms, we find that for a photon emitted from the point $\underset{\sim}{x}_1$ in an initial direction $\underset{\sim}{\hat{n}}$,

$$
\begin{aligned}
\hat{\mathbf{K}}(\mathbf{x}) = {} & (1+\gamma)\frac{M}{d^2}\left[\frac{\hat{\mathbf{n}}(\mathbf{x}\cdot\mathbf{x}_1)}{r} - \frac{\mathbf{x}_1(\hat{\mathbf{n}}\cdot\mathbf{x})}{r} + \frac{\mathbf{d}(\hat{\mathbf{n}}\cdot\mathbf{x}_1)}{r_1}\right] \\
& - \frac{1}{4}\Delta\left\{\frac{(\mathbf{x}_1\times\mathbf{J})}{r_1^3} + \frac{(\hat{\mathbf{n}}\times\mathbf{J})}{d^2}\left[\frac{(\hat{\mathbf{n}}\cdot\mathbf{x})}{r} - \frac{(\hat{\mathbf{n}}\cdot\mathbf{x}_1)}{r_1}\right]\right. \\
& \qquad + \hat{\mathbf{n}}\left[(\hat{\mathbf{n}}\times\mathbf{J})\cdot\mathbf{x}_1\right]\left(\frac{1}{r^3} - \frac{1}{r_1^3}\right) \\
& \left. + \left[\hat{\mathbf{n}}\times(\hat{\mathbf{n}}\times\mathbf{x}_1)\right]\left[(\hat{\mathbf{n}}\times\mathbf{J})\cdot\mathbf{x}_1\right]\left[\left(\frac{\hat{\mathbf{n}}\cdot\mathbf{x}}{r^3} - \frac{\hat{\mathbf{n}}\cdot\mathbf{x}_1}{r_1^3}\right) + \frac{2}{d^2}\left(\frac{\hat{\mathbf{n}}\cdot\mathbf{x}}{r} - \frac{\hat{\mathbf{n}}\cdot\mathbf{x}_1}{r_1}\right)\right]\right\} \\
& + \frac{1}{2}(\alpha_1 - 2\alpha_2)M\left[\frac{\mathbf{w}}{r_1} - \frac{\hat{\mathbf{n}}(\mathbf{x}\cdot\mathbf{x}_1)}{d^2 r} + \frac{\mathbf{x}_1(\hat{\mathbf{n}}\cdot\mathbf{x})}{d^2 r} - \frac{\mathbf{d}(\hat{\mathbf{n}}\cdot\mathbf{x}_1)}{d^2 r_1}\right] \\
& + \alpha_2 M\left\{\frac{(\mathbf{x}_1\cdot\mathbf{w})\mathbf{x}_1}{r_1^3} + \frac{\hat{\mathbf{n}}}{d^2}\left[\frac{(\hat{\mathbf{n}}\cdot\mathbf{w})(\mathbf{x}\cdot\mathbf{x}_1)}{r} - \frac{(\mathbf{w}\cdot\mathbf{x}_1)(\hat{\mathbf{n}}\cdot\mathbf{x})}{r} + \frac{(\mathbf{w}\cdot\mathbf{d})(\hat{\mathbf{n}}\cdot\mathbf{x}_1)}{r_1}\right]\right. \\
& \qquad + \frac{\mathbf{w}}{d^2}\left[\frac{\mathbf{x}\cdot\mathbf{x}_1}{r} - \frac{(\hat{\mathbf{n}}\cdot\mathbf{x}_1)(\hat{\mathbf{n}}\cdot\mathbf{x})}{r}\right] \\
& - (\hat{\mathbf{n}}\cdot\mathbf{w})\hat{\mathbf{n}}\left[\frac{3}{r} - \frac{r_1^2}{r^3} - \frac{2}{r_1} - \frac{2}{r}(\hat{\mathbf{n}}\cdot\mathbf{x})^2\left(\frac{1}{r^2} - \frac{1}{d^2}\right) + \frac{2}{r_1}(\hat{\mathbf{n}}\cdot\mathbf{x}_1)^2\left(\frac{1}{r_1^2} - \frac{1}{d^2}\right)\right. \\
& \left. - \frac{2}{r^3}(\hat{\mathbf{n}}\cdot\mathbf{x})^2 + \frac{2}{r_1^3}(\hat{\mathbf{n}}\cdot\mathbf{x}_1)^2 - \frac{2}{d^2 r}(\hat{\mathbf{n}}\cdot\mathbf{x})^3(\hat{\mathbf{n}}\cdot\mathbf{x}_1)\left(\frac{1}{r^2} + \frac{2}{d^2}\right) + \frac{2}{d^2 r_1}(\hat{\mathbf{n}}\cdot\mathbf{x}_1)^4\left(\frac{1}{r_1^2} + \frac{2}{d^2}\right)\right] \\
& - \left[\frac{\hat{\mathbf{n}}\cdot\mathbf{x}}{r}\left(\frac{1}{r^2} - \frac{1}{d^2}\right) - \frac{\hat{\mathbf{n}}\cdot\mathbf{x}_1}{r_1}\left(\frac{1}{r_1^2} - \frac{1}{d^2}\right) - \frac{2}{r^3}(\hat{\mathbf{n}}\cdot\mathbf{x}_1) + \frac{2}{r_1^3}(\hat{\mathbf{n}}\cdot\mathbf{x}_1)\right. \\
& \left. - \frac{1}{d^2 r}(\hat{\mathbf{n}}\cdot\mathbf{x})(\hat{\mathbf{n}}\cdot\mathbf{x}_1)^2\left(\frac{1}{r^2} + \frac{2}{d^2}\right) + \frac{1}{d^2 r_1}(\hat{\mathbf{n}}\cdot\mathbf{x}_1)^3\left(\frac{1}{r_1^2} + \frac{2}{d^2}\right)\right]\left[(\hat{\mathbf{n}}\cdot\mathbf{w})\mathbf{x}_1 + (\mathbf{x}_1\cdot\mathbf{w})\hat{\mathbf{n}} + (\mathbf{x}_1\cdot\hat{\mathbf{n}})(\hat{\mathbf{n}}\cdot\mathbf{w})\hat{\mathbf{n}}\right] \\
& - \left[\frac{1}{r^3} - \frac{1}{r_1^3} + \frac{(\hat{\mathbf{n}}\cdot\mathbf{x})(\hat{\mathbf{n}}\cdot\mathbf{x}_1)}{d^2 r}\left(\frac{1}{r^2} + \frac{2}{d^2}\right) - \frac{(\hat{\mathbf{n}}\cdot\mathbf{x}_1)^2}{d^2 r_1}\left(\frac{1}{r_1^2} + \frac{2}{d^2}\right)\right]\left[(\mathbf{x}_1\cdot\mathbf{w})\mathbf{x}_1 + (\hat{\mathbf{n}}\cdot\mathbf{w})(\hat{\mathbf{n}}\cdot\mathbf{x}_1)\mathbf{x}_1 + (\hat{\mathbf{n}}\cdot\mathbf{x}_1)(\mathbf{x}_1\cdot\mathbf{w})\hat{\mathbf{n}}\right] \\
& \left. + \left[\frac{\hat{\mathbf{n}}\cdot\mathbf{x}}{d^2 r}\left(\frac{1}{r^2} + \frac{2}{d^2}\right) - \frac{\hat{\mathbf{n}}\cdot\mathbf{x}_1}{d^2 r_1}\left(\frac{1}{r_1^2} + \frac{2}{d^2}\right)\right](\hat{\mathbf{n}}\cdot\mathbf{x}_1)(\mathbf{w}\cdot\mathbf{x}_1)\mathbf{x}_1\right\}, \qquad (3.7)
\end{aligned}
$$

where,

$$\mathbf{d} = (\hat{\mathbf{n}}\times\mathbf{x}_1).$$

IV. Final Results

We now have all of the ingredients necessary to calculate the relativistic frequency shift to order c^{-4}. Using Eq. (2.8) in Eq. (2.4) yields

$$
\begin{aligned}
\frac{f(\mathbf{x}_2)}{f(\mathbf{x}_1)} = 1 &- \hat{\mathbf{n}}\cdot(\mathbf{v}_2 - \mathbf{v}_1) - (U_1 - U_2) - \tfrac{1}{2}(v_1^2 - v_2^2) \\
&- \left[(\hat{\mathbf{n}}\cdot\mathbf{v}_1) + (\hat{\mathbf{n}}\cdot\mathbf{v}_1)^2 + (\hat{\mathbf{n}}\cdot\mathbf{v}_1)^3\right]\hat{\mathbf{n}}\cdot(\mathbf{v}_2 - \mathbf{v}_1) \\
&- \left[\hat{\mathbf{n}}\cdot(\mathbf{v}_2 - \mathbf{v}_1) + (\hat{\mathbf{n}}\cdot\mathbf{v}_1)^2\right]\left[(U_1 - U_2) + \tfrac{1}{2}(v_1^2 - v_2^2)\right] \\
&- \left[(\hat{\mathbf{k}}_2\cdot\mathbf{v}_2) - (\hat{\mathbf{k}}_1\cdot\mathbf{v}_1)\right] + 2(\hat{\mathbf{n}}\cdot\mathbf{v}_1)(\hat{\mathbf{k}}_1\cdot\mathbf{v}_1) \\
&\qquad - (\hat{\mathbf{n}}\cdot\mathbf{v}_1)(\hat{\mathbf{k}}_2\cdot\mathbf{v}_2) - (\hat{\mathbf{n}}\cdot\mathbf{v}_2)(\hat{\mathbf{k}}\cdot\mathbf{v}_1) \\
&- \tfrac{3}{8}(v_1^4 - v_2^4) - \tfrac{1}{4}(v_1^2 v_2^2 - v_1^4) \\
&- \left[\left(\tfrac{3}{2} - \beta\right)(U_1^2 - U_2^2) - (U_1^2 - U_1 U_2)\right] \\
&- \left[\left(\tfrac{3}{2} + \gamma\right)(U_1 v_1^2 - U_2 v_2^2) + \tfrac{1}{2}(U_1 v_2^2 + U_2 v_1^2 - 2U_1 v_1^2)\right] \\
&+ \tfrac{1}{4}\Big\{\Delta\left[\mathbf{v}_1\cdot(\hat{\mathbf{r}}_1\times\mathbf{J})/r_1^2 - \mathbf{v}_2\cdot(\hat{\mathbf{r}}_2\times\mathbf{J})/r_2^2\right] \\
&\qquad + (\alpha_1 - 2\alpha_3)\,\mathbf{w}\cdot\left[(\hat{\mathbf{r}}_1\times\mathbf{J})/r_1^2 - (\hat{\mathbf{r}}_2\times\mathbf{J})/r_2^2\right]\Big\} \\
&- \tfrac{1}{2}\Big\{(\alpha_1 - \alpha_2 - \alpha_3)\,w^2(U_1 - U_2) + \alpha_2\left[U_1(\mathbf{w}\cdot\hat{\mathbf{r}}_1)^2 - U_2(\mathbf{w}\cdot\hat{\mathbf{r}}_2)^2\right] \\
&\qquad + (\alpha_1 - 2\alpha_2)\left[(\mathbf{v}_1\cdot\mathbf{w})U_1 - (\mathbf{v}_2\cdot\mathbf{w})U_2\right] \\
&\qquad + 2\alpha_2\left[U_1(\mathbf{v}_1\cdot\hat{\mathbf{r}}_1)(\mathbf{w}\cdot\hat{\mathbf{r}}_1) - U_2(\mathbf{v}_2\cdot\hat{\mathbf{r}}_2)(\mathbf{w}\cdot\hat{\mathbf{r}}_2)\right]\Big\} \qquad (4.1
\end{aligned}
$$

where $\mathbf{k}(\mathbf{x})$ is given by Eq. (3.7).

A possible limitation of Eq. (4.1) is the assumption of the validity of the EEP. We need to incorporate into Eq. (4.1) possible violations of local Lorentz invariance (LLI) or local position invariance (LPI). To be rigorous, we should adopt a complete nonmetric formalism which reduces to the PPN formalism in an appropriate limit.[9] We have avoided this complication in this paper, having focussed instead upon the standard PPN formalism.[10] It is possible, however, to generalize to a certain extent the results already at hand.

We can account for certain possible violations by inserting the additional parameters $(\tilde{\alpha}, \epsilon_1, \epsilon_2)$ into Eq. (2.7) such that

$$u^0 = 1 + \tilde{\alpha} U + \frac{1}{2}\epsilon_1 v^2 + \frac{3}{8}\epsilon_2 v^4 + ..., \tag{4.2}$$

The PPN parameters β and γ in Eq. (2.7) should be relabeled in order to absorb other possible LLI or LPI violating terms having similar dependencies. We accomplish this by simply placing a "tilde" over the parameters, but leave unchanged the meanings of the gravitomagnetic parameter Δ and the preferred-frame parameters α_1 and α_2. Violations of LLI or LPI could have a different affect on the photon equation of motion. Therefore, to carry our generalization further, the parameter γ in Eq. (3.7) should be given still a different name, perhaps by giving it a "p" subscript. With this in mind, our generalization of Eq. (4.1) is given by

$$
\begin{aligned}
\frac{f(\mathbf{x}_2)}{f(\mathbf{x}_1)} = {} & 1 - \hat{\mathbf{n}}\cdot(\mathbf{x}_2 - \mathbf{x}_1) - \tilde{\alpha}\,(U_1 - U_2) - \frac{\epsilon_1}{2}\left(v_1^2 - v_2^2\right) \\
& - \left[(\hat{\mathbf{n}}\cdot\mathbf{x}_1) + (\hat{\mathbf{n}}\cdot\mathbf{x}_1)^2 + (\hat{\mathbf{n}}\cdot\mathbf{x}_1)^3\right]\hat{\mathbf{n}}\cdot(\mathbf{x}_2 - \mathbf{x}_1) \\
& - \left[\hat{\mathbf{n}}\cdot(\mathbf{x}_2 - \mathbf{x}_1) + (\hat{\mathbf{n}}\cdot\mathbf{x}_1)^2\right]\left[\tilde{\alpha}\,(U_1 - U_2) + \frac{\epsilon_1}{2}\left(v_1^2 - v_2^2\right)\right] \\
& - \left[(\hat{\mathbf{k}}_2\cdot\mathbf{x}_2) - (\hat{\mathbf{k}}_1\cdot\mathbf{x}_1)\right] + 2\,(\hat{\mathbf{n}}\cdot\mathbf{x}_1)(\hat{\mathbf{k}}_1\cdot\mathbf{x}_1) - (\hat{\mathbf{n}}\cdot\mathbf{x}_1)(\hat{\mathbf{k}}_2\cdot\mathbf{x}_2) - (\hat{\mathbf{n}}\cdot\mathbf{x}_2)(\hat{\mathbf{k}}\cdot\mathbf{x}_1) \\
& - \epsilon_2\,\frac{3}{8}\left(v_1^4 - v_2^4\right) - \frac{\epsilon_1^2}{4}\left(v_1^2 v_2^2 - v_1^4\right) \\
& - \left[\left(\frac{3}{2} - \tilde{\beta}\right)\left(U_1^2 - U_2^2\right) - \tilde{\alpha}^2\left(U_1^2 - U_1 U_2\right)\right] \\
& - \left[\left(\frac{3}{2} + \tilde{\gamma}\right)\left(U_1 v_1^2 - U_2 v_2^2\right) + \frac{\epsilon_1\tilde{\alpha}}{2}\left(U_1 v_2^2 + U_2 v_1^2 - 2U_1 v_1^2\right)\right] \\
& + \frac{1}{4}\Big\{\Delta\left[\mathbf{x}_1\cdot(\hat{\mathbf{r}}_1\times\mathbf{J})/r_1^2 - \mathbf{x}_2\cdot(\hat{\mathbf{r}}_2\times\mathbf{J})/r_2^2\right] \\
& \qquad + (\alpha_1 - 2\alpha_3)\,\mathbf{w}\cdot\left[(\hat{\mathbf{r}}_1\times\mathbf{J})/r_1^2 - (\hat{\mathbf{r}}_2\times\mathbf{J})/r_2^2\right]\Big\} \\
& - \frac{1}{2}\Big\{(\alpha_1 - \alpha_2 - \alpha_3)\,w^2\,(U_1 - U_2) + \alpha_2\left[U_1(\mathbf{w}\cdot\hat{\mathbf{r}}_1)^2 - U_2(\mathbf{w}\cdot\hat{\mathbf{r}}_2)^2\right] \\
& \qquad + (\alpha_1 - 2\alpha_2)\left[(\mathbf{x}_1\cdot\mathbf{w})U_1 - (\mathbf{x}_2\cdot\mathbf{w})U_2\right] \\
& \qquad + 2\alpha_2\left[U_1(\mathbf{x}_1\cdot\hat{\mathbf{r}}_1)(\mathbf{w}\cdot\hat{\mathbf{r}}_1) - U_2(\mathbf{x}_2\cdot\hat{\mathbf{r}}_2)(\mathbf{w}\cdot\hat{\mathbf{r}}_2)\right]\Big\}
\end{aligned}
\tag{4.3}
$$

V. Conclusions

We have presented a derivation of the gravitational redshift effect to order c^{-4}. The calculation was performed within the framework of the PPN formalism for metric theories of gravity, but we also considered how to include possible violations of the equivalence principle. As mentioned in the Introduction, our primary motivation was to be able to model accurately a possible solar probe test of the redshift. If an atomic frequency standard, such as a hydrogen maser, were flown on the spacecraft, then it might be possible to detect fractional frequency variations as small as 1 part in 10^{16}. At the periapsis of 4 solar radii, we would be sensitive to first-order effects of the Newtonian potential U as small as 2 parts in 10^{10}, to second-order effects of U^2 of 4 parts in 10^4, and to preferred-frame effects of order w^2U of 1 part in 10^4. The gravitomagnetic effect arising from solar rotation of order vJ/r could produce a fractional frequency shift of roughly 5×10^{-16} at periapsis, and thus might be detectable. Currently, we are planning to use Eq. (4.1), along with the PPN equation of motion of the spacecraft, in a rigorous covariance analysis of the planned mission to determine more precisely the sensitivity of the experiment to these second-order effects.

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10. We also defer including in our present analysis the nonsymmetric gravitational theory of Moffat, which is not fully described by the standard PPN formalism.

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